

INFINITELY MANY SOLUTIONS FOR THE DIRICHLET PROBLEM INVOLVING THE P-LAPLACIAN IN ANNULUS

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ABSTRACT. We present a result of existence of infinitely many solutions for the Dirichlet problem involving the p-Laplacian in annular domains, when $p \leq N$, contouring the failure of compactness of $W^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ applying a variable change.

1. INTRODUCTION

This paper is to investigate the following autonomous Dirichlet problem in annulus

$$(1.1) \quad \begin{cases} -\Delta_p u = f(u) & \text{in } \Omega_{a,b}, \\ u = 0 & \text{on } \partial\Omega_{a,b} \end{cases}$$

where $\Omega_{a,b} = \{x \in \mathbb{R}^N : a < |x| < b\}$ with $0 < a < b$ constants in \mathbb{R} ,

$$1 < p \leq N,$$

$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

In order to study the solution of (1.1), one can make a standard change of variables. In the $N > p$, be $t = -\frac{A}{r^{(N-p)/(p-1)}} + B$ and $v(t) = u(r)$, where

$$A = \frac{(ab)^{\frac{N-p}{p-1}}}{b^{\frac{N-p}{p-1}} - a^{\frac{N-p}{p-1}}} \quad \text{and} \quad B = \frac{b^{\frac{N-p}{p-1}}}{b^{\frac{N-p}{p-1}} - a^{\frac{N-p}{p-1}}},$$

then the problem (1.1) transforms into the boundary value problem for the nonautonomous ODE

$$(1.2) \quad \begin{cases} (|v'(t)|^{p-2} v'(t))' + q(t) f(v(t)) = 0 & \text{in } (0, 1), \\ v(0) = v(1) = 0, \end{cases}$$

where

$$(1.3) \quad q(t) = \left(\frac{p-1}{N-p} \right)^p \frac{A^{\frac{(p-1)p}{N-p}}}{(B-t)^{\frac{p(N-1)}{N-p}}}.$$

In the case $p = N$, one sets $r = a(\frac{b}{a})^t$ and $v(t) = u(r)$, obtaining again the problem (1.2), now with

$$q(t) = \left[a \left(\frac{b}{a} \right)^t \ln \frac{b}{a} \right]^p.$$

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Note that in both cases, the function $q(t)$ is well defined, continuous and bounded between positive constants in the interval $[0, 1]$, that is, there exist $q_1, q_0 > 0$ such that

$$0 < q_0 \leq q(t) \leq q_1.$$

For our purpose, we shall restrict our attention to the ordinary boundary value problem (1.2), where the function $q(t)$ is continuous and positive on the interval $[0, 1]$, while for f we consider the assumptions below. A weak solution of (1.2) is any $v \in W_0^{1,p}(0, 1)$ such that

$$\int_0^1 |v'(t)|^{p-2} v'(t) w'(t) dt - \int_0^1 q(t) f(v(t)) w(t) dt = 0,$$

for each $w \in W_0^{1,p}(0, 1)$.

We are interested in the existence of infinitely many non-negative weak solutions for problem (1.1), or equivalently, to the problem (1.2). Precisely, if $F(\xi) = \int_0^\xi f(t) dt$, our aim is to prove the following results:

Theorem 1.1. *Assume that $f(0) = 0$, $q(t) \geq q_0 > 0$ in $[0, 1]$ and $\inf_{\xi \geq 0} F(\xi) \geq 0$. Moreover, suppose that there exist two sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$ in $]0, +\infty[$, with $a_k < b_k$, $\lim_{k \rightarrow +\infty} b_k = +\infty$, such that*

- (i): $\lim_{k \rightarrow +\infty} \frac{b_k}{a_k} = +\infty$;
- (ii): $\max_{[a_k, b_k]} f \leq 0$ for all $k \in \mathbb{N}$;
- (iii): Suppose that

$$\frac{\sigma(p, q_0)}{p(\sup_{t \in (0,1)} \text{dist}(t, \{0, 1\}))^p} < \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^p} < +\infty,$$

where

$$\sigma(p, q_0) = \inf_{\mu \in]0, 1[} \frac{1}{q_0 \mu (1 - \mu)^{p-1}}.$$

Then, problem (1.2) (respectively the problem (1.1)) admits an unbounded sequence of non-negative weak solutions in $W_0^{1,p}(0, 1)$ (respectively in $W_0^{1,p}(\Omega_{a,b})$).

Remark 1.2. The constant $\sigma(p, q_0)$ is well defined. To see this, let $\sigma : (0, 1) \rightarrow \mathbb{R}$ be the function defined by $\sigma(x) = \frac{1}{x(1-x)^{p-1}}$. As $\sigma'(x) = \frac{-(1-x)^{p-2}(1-px)}{x^2(1-x)^{2(p-1)}}$ we have $\sigma'(x) < 0$ if $0 < x < \frac{1}{p}$, $\sigma'(\frac{1}{p}) = 0$ and $\sigma'(x) > 0$ if $\frac{1}{p} < x < 1$. Therefore,

$$\sigma(p, q_0) = \inf_{\mu \in]0, 1[} \frac{1}{q_0 \mu (1 - \mu)^{p-1}} = \frac{1}{q_0 p}.$$

Theorem 1.3. *Assume that $f(0) = 0$, $q(t) \geq q_0 > 0$ in $[0, 1]$ and $\inf_{\xi \geq 0} F(\xi) \geq 0$. Moreover, suppose that there exist two sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$ in $]0, +\infty[$, with $a_k < b_k$, $\lim_{k \rightarrow +\infty} b_k = +\infty$, such that*

- (i): $\lim_{k \rightarrow +\infty} \frac{b_k}{a_k} = +\infty$;
- (ii): $\max_{[a_k, b_k]} f \leq 0$ for all $k \in \mathbb{N}$;
- (iii): Suppose that

$$\frac{\sigma(p, q_0)}{p(\sup_{t \in (0,1)} \text{dist}(t, \{0, 1\}))^p} < \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^p} < +\infty,$$

where

$$\sigma(p, q_0) = \inf_{\mu \in]0, 1[} \frac{1}{q_0 \mu (1 - \mu)^{p-1}}.$$

Then, problem (1.2) (respectively the problem (1.1)) admits an sequence of non-negative weak solutions, which strongly converges to 0 in $W_0^{1,p}(0,1)$ (respectively in $W_0^{1,p}(\Omega_{a,b})$).

The statement and the proves of Theorems 1.1 and 1.3 are very similar the Theorems 1.1 and 1.2 of F. Cammaroto, A. Chinn, B. Di Bella [2] which is the principal motivation and reference of this paper. But in [2] the authors supposed that $p > N$ and a standard argument, chiefly based on the compact embedding $W^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ while in this paper we consider $p \leq N$ and soon, the above mentioned immersion is not satisfied. For the sake of completeness, we decided to remake the proves of Theorems 1.1 and 1.3 to explicit the most important changes in specific arguments compared with [2].

The existence of infinitely many solutions of problem (1.1) in general bounded domains Ω has been studied extensively. Among them, the ones which are closest to the present article are certainly [2, 3, 4] and references therein. The approach used in [2] is based on a recent variational principle obtained by Ricceri in [6], while in [3, 4] is based on the method of lower and upper solutions.

A more general problem that (1.1) was studied in Bonanno and Bisci [1] in general bounded domains Ω , and the authors also supposed that $p > N$ following the same arguments of [2]. Also examples and applications are given in comparison with [2]. Following ours results, we can prove analogous results to the results of [1], in $\Omega_{a,b}$, when $p \leq N$.

The ours approach is based on the changed of variables, problem (1.2), and the recent variational principle obtained by Ricceri in [6]. The following result is a direct consequence of Theorem 2.5 of [6].

Proposition 1.4. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functional. Assume also that Ψ is (strongly) continuous and satisfies $\lim_{\|x\| \rightarrow +\infty} \Psi(x) = +\infty$. For each $r > \inf_X \Psi$, put*

$$\varphi(r) = \inf_{x \in \Psi^{-1}([-\infty, r])} \frac{\Phi(x) - \inf_{(\Psi^{-1}([-\infty, r]))_w} \Phi}{r - \Psi(x)},$$

where $(\Psi^{-1}([-\infty, r]))_w$ is the closure of $\Psi^{-1}([-\infty, r])$ in the weak topology. Fixed $\lambda \in \mathbb{R}$, then

- (1): if $\{r_n\}_{n \in \mathbb{N}}$ is a real sequences with $\lim_{n \rightarrow +\infty} r_n = +\infty$ such that $\varphi(r_n) < \lambda$, for each $n \in \mathbb{N}$, the following alternative holds: either $\Phi + \lambda\Psi$ has a global minimum, or there exists a sequence $\{x_n\}$ of critical points of $\Phi + \lambda\Psi$ such that $\lim_{n \rightarrow +\infty} \Psi(x_n) = +\infty$.
- (2): if $\{s_n\}_{n \in \mathbb{N}}$ is a real sequences with $\lim_{n \rightarrow +\infty} s_n = (\inf_X \Psi)^+$ such that $\varphi(s_n) < \lambda$, for each $n \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$, or there exists a sequence $\{x_n\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{n \rightarrow +\infty} \Psi(x_n) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ .

Before stating our main result, we wish to point out that in the sequel $f(x) = 0$ for each $x \in]-\infty, 0[$. We shall consider the Sobolev space $W_0^{1,p}(0,1)$ endowed with

the norm

$$\|v\| := \left(\int_0^1 |v'(t)|^p dt \right)^{1/p}.$$

We recall that there exists a constant $c > 0$ such that

$$(1.4) \quad \sup_{x \in [0,1]} |v(t)| \leq c\|v\|$$

for each $v \in W_0^{1,p}(0,1)$.

2. PROOFS

Proof of Theorem 1.1. Let us apply part (a) of Proposition 1.4. To this end choose $X = W_0^{1,p}(0,1)$ and for each $v \in X$, put

$$\Phi(v) = - \int_0^1 \left(\int_0^{u(t)} q(t)f(s)ds \right) dt$$

and

$$\Psi(v) = \|v\|^p.$$

It is well known that the critical points in X of the functional $\Phi + (1/p)\Psi$ are precisely the weak solutions of problem (1.2). By the compact embedding of $W_0^{1,p}(0,1)$ in $C([0,1])$, it is not difficult ensures that the functionals Φ and Ψ are Gâteaux differentiable and sequentially weakly lower semicontinuous, moreover Ψ is obviously (strong) continuous and coercive.

In our case the function φ of Proposition 1.4 is defined by setting

$$\varphi(r) = \inf_{\|v\|^p < r} \frac{\sup_{\|v\|^p < r} \int_0^1 q(t)F(v(t))dt - \int_0^1 q(t)F(v(t))dt}{r - \|v\|^p}$$

for each $r \in]0, +\infty[$.

Now, put $r_k = \left(\frac{b_k}{c}\right)^p$, we wish to prove that $\varphi(r_k) < \frac{1}{p}$ for each $k \in \mathbb{N}$. To this aim, it suffices to prove that, for each $k \in \mathbb{N}$, there exists a function $v_k \in X$, with $\|v_k\|^p < r_k$, such that

$$\sup_{\|v\|^p < r_k} \int_0^1 q(t)F(v(t))dt - \int_0^1 q(t)F(v_k(t))dt < \frac{1}{p}(r_k - \|v_k\|^p).$$

From (iii) we can choose a constant h such that

$$(2.1) \quad \frac{\sigma(p, q_0)}{p(\sup_{t \in (0,1)} \text{dist}(t, \{0,1\}))^p} < h < \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^p}$$

and so there exists $t_0 \in (0,1)$ such that $\left(\frac{\sigma(p, q_0)}{ph}\right)^{1/p} < \text{dist}(t_0, \{0,1\})$. Therefore, we can fix γ satisfying

$$\left(\frac{\sigma(p, q_0)}{ph}\right)^{1/p} < \gamma < \text{dist}(t_0, \{0,1\}).$$

Observe that by (ii),

$$(2.2) \quad \max_{[0, a_k]} F = \max_{[0, b_k]} F.$$

Now, fix $k \in \mathbb{N}$ and consider the function $v_k \in X$ defined by setting

$$v_k(t) = \begin{cases} 0 & \text{if } t \in (0, 1) \setminus (t_0 - \gamma, t_0 + \gamma), \\ \xi_k & \text{if } t \in (t_0 - \frac{\gamma}{2}, t_0 + \frac{\gamma}{2}), \\ \frac{2\xi_k}{\gamma}(\gamma - |t - t_0|) & \text{if } t \in (t_0 - \gamma, t_0 + \gamma) \setminus (t_0 - \frac{\gamma}{2}, t_0 + \frac{\gamma}{2}) \end{cases}$$

with $\xi_k \in]0, a_k]$ such that

$$(2.3) \quad F(\xi_k) = \max_{\eta \in [0, a_k]} F(\eta).$$

In view of (i), we can choose $k \in \mathbb{N}$ so that

$$(2.4) \quad \frac{b_k}{a_k} > \frac{2c}{\gamma} (\gamma)^{1/p}$$

for all $k > \bar{k}$.

Fix $k > \bar{k}$. By (2.4) we have

$$\begin{aligned} \|v_k\|^p &= \int_0^1 |v'_k(t)|^p dt = \int_{(t_0 - \gamma, t_0 + \gamma) \setminus (t_0 - \frac{\gamma}{2}, t_0 + \frac{\gamma}{2})} \frac{2^p \xi_k^p}{\gamma^p} dt \\ &= \frac{2^p \xi_k^p}{\gamma^p} (|(t_0 - \gamma, t_0 + \gamma)| - |(t_0 - \frac{\gamma}{2}, t_0 + \frac{\gamma}{2})|) = \frac{2^p \xi_k^p}{\gamma^p} (2\gamma - \gamma) \\ &= \frac{2^p \xi_k^p}{\gamma^{p-1}} < r_k. \end{aligned}$$

Because of (1.4), (2.2) and (2.3), for each $v \in X$, fulfilling $\|v\|^p \leq r_k$, one has

$$\sup_{x \in [0, 1]} |v(t)| \leq cr_k^{1/p}$$

as well as

$$F(v(t)) \leq \max_{\eta \in [0, cr_k^{1/p}]} F(\eta) = F(\xi_k) \text{ in } (0, 1).$$

Next, since $\lim_{k \rightarrow +\infty} \frac{r_k}{\xi_k^p} = +\infty$, there exists $k_0 \in \mathbb{N}$ such that

$$(2.5) \quad \frac{r_k}{\xi_k^p} > p \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^p} \left(\int_0^1 q(t) dt - \int_{t_0 - \frac{\gamma}{2}}^{t_0 + \frac{\gamma}{2}} q(t) dt \right) + \frac{2^p}{\gamma^{p-1}}$$

for all $k > k_0$. Hence, using (2.5), we get

$$\begin{aligned} &\sup_{\|v\|^p < r_k} \int_0^1 q(t) F(v(t)) dt - \int_0^1 q(t) F(v_k(t)) dt \\ &\leq F(\xi_k) \int_0^1 q(t) dt - \int_{t_0 - \frac{\gamma}{2}}^{t_0 + \frac{\gamma}{2}} q(t) F(\xi_k) dt \\ &\leq F(\xi_k) \left(\int_0^1 q(t) dt - \int_{t_0 - \frac{\gamma}{2}}^{t_0 + \frac{\gamma}{2}} q(t) dt \right) \\ &\leq \frac{1}{p} (r_k - \|v_k\|^p) \end{aligned}$$

for each $k > k^* \geq k_0$.

Since $\lim_{k \rightarrow +\infty} r_k = +\infty$, the previous inequality assures that the conclusion (1) of Proposition 1.4 can be used and either the functional $\Phi + (1/p)\Psi$ has a global minimum, or there exists a sequence $\{v_k\}_{k \in \mathbb{N}}$ of solutions of problem (1.2) such that $\lim_{k \rightarrow +\infty} \|v_k\| = +\infty$.

The other step is to verify that the functional $\Phi + (1/p)\Psi$ has no global minimum. Taking into account (2.1), one has, for each $k \in \mathbb{N}$.

$$\sup_{\eta \geq k} \frac{F(\eta)}{\eta^p} > h$$

and so there exists $\eta_k \geq k$ such that

$$\frac{F(\eta_k)}{\eta_k^p} > h.$$

Now, if we consider a function $w_k \in X$ defined by setting

$$w_k(t) = \begin{cases} 0 & \text{if } x \in (0, 1) \setminus (t_0 - \gamma, t_0 + \gamma), \\ \eta_k & \text{if } x \in (t_0 - \bar{\mu}\gamma, t_0 + \bar{\mu}\gamma), \\ \frac{\eta_k}{\gamma(1 - \bar{\mu})}(\gamma - |t - t_0|) & \text{if } x \in (t_0 - \gamma, t_0 + \gamma) \setminus (t_0 - \bar{\mu}\gamma, t_0 + \bar{\mu}\gamma) \end{cases}$$

where $\bar{\mu} \in]0, 1[$ is such that

$$\sigma(p, q_0) = \frac{1}{q_0 \bar{\mu} (1 - \bar{\mu})^{p-1}},$$

note that $\bar{\mu}$ is well defined by Remark 1.2. We have

$$\begin{aligned} & \Phi(w_k) + \frac{1}{p}\Psi(w_k) \\ &= - \int_0^1 q(t)F(w_k(t))dt + \frac{1}{p}\|w_k\|^p \\ &\leq - \int_{t_0 - \bar{\mu}\gamma}^{t_0 + \bar{\mu}\gamma} q(t)F(\eta_k)dt + \frac{1}{p} \frac{\eta_k^p}{\gamma^p (1 - \bar{\mu})^p} (2\gamma - 2\bar{\mu}\gamma) \\ &\leq -2\bar{\mu}\gamma q_0 F(\eta_k) + \frac{2\gamma}{p} \frac{\eta_k^p}{\gamma^p (1 - \bar{\mu})^{p-1}} \\ &= 2\bar{\mu}\gamma q_0 \left(\frac{1}{p} \frac{\eta_k^p}{\gamma^p q_0 \bar{\mu} (1 - \bar{\mu})^{p-1}} - F(\eta_k) \right) \\ &< 2\bar{\mu}\gamma q_0 \left(\frac{1}{p} \frac{\eta_k^p}{\gamma^p q_0 \bar{\mu} (1 - \bar{\mu})^{p-1}} - h \eta_k^p \right) = 2\bar{\mu}\gamma q_0 \eta_k^p \left(\frac{\sigma(p, q_0)}{p \gamma^p} - h \right). \end{aligned}$$

Since $h > \frac{\sigma(p, q_0)}{p \gamma^p}$, we conclude that $\lim_{k \rightarrow +\infty} \eta_k^p \left(\frac{\sigma(p, q_0)}{p \gamma^p} - h \right) = -\infty$ and so the previous inequality shows that the functional $\Phi + (1/p)\Psi$ is not bounded from below and then it has no global minimum.

Therefore, Proposition 1.4 assures that there is a sequence $\{v_k\}_{k \in \mathbb{N}} \subset X$ of critical points of $\Phi + (1/p)\Psi$ such that $\lim_{n \rightarrow +\infty} \|v_n\| = +\infty$. As previously observed, every function v_k is a weak solution of (1.2).

Finally, we claim that each weak solution of problem (1.2) is non-negative in $(0, 1)$. Assume the contrary. Let v be a weak solution of (1.2) such that the set $A = \{x \in (0, 1) : v(x) < 0\}$ is non-empty. By the continuity of v , A is open and so $v|_A \in W_0^{1,p}(A)$. Then, for each $w \in W_0^{1,p}(A)$,

$$\int_A |v'(t)|^{p-2} v'(t) w'(t) dt - \int_A q(t) f(v(t)) w(t) dt = 0.$$

The assumptions on f imply

$$\int_A |v'(t)|^{p-2} v'(t) w'(t) dt = 0$$

for each $w \in W_0^{1,p}(A)$ and so, in particular, one has

$$\int_A |v'(t)|^p dt = 0,$$

an absurd. This completes the proof. \square

Proof of Theorem 1.3. We take X, Φ, Ψ as in the proof of Theorem 1.1. In a similar way we prove that $\varphi(s_k) < \frac{1}{p}$ for each $k \in \mathbb{N}$, with $s_k = (\frac{b_k}{c})^p$. Now, fixed h such that

$$\frac{\sigma(p, q_0)}{p(\sup_{t \in (0,1)} \text{dist}(t, \{0, 1\}))^p} < h < \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^p}.$$

For each $k \in \mathbb{N}$, there exists $\eta_k \leq \frac{1}{k}$ such that $\frac{F(\eta_k)}{\eta_k^p} > h$. If we take w_k as in the proof of Theorem 1.1, of course the sequence $\{w_k\}$ strongly converges to 0 in X and $\Phi(w_k) + (1/p)\Psi(w_k) < 0$ for all $k \in \mathbb{N}$. Since $\Phi(0) + (1/p)\Psi(0) = 0$, this means that 0 is not a local minimum of $\Phi + (1/p)\Psi$. Then, since 0 is the only a global minimum of Ψ , the part (2) of Proposition 1.4 ensures that there exists a sequence $\{v_k\}_{k \in \mathbb{N}} \subset X$ of critical points of $\Phi + (1/p)\Psi$ such that $\lim_{k \rightarrow +\infty} \|v_k\| = 0$ and this completes the proof. \square

3. CONCLUSIONS

We can also to investigate the following nonautonomous Dirichlet problem in an annular domain

$$(3.1) \quad \begin{cases} -\Delta_p u = g(|x|)f(u) & \text{in } \Omega_{a,b}, \\ u = 0 & \text{on } \partial\Omega_{a,b} \end{cases}$$

where $\Omega_{a,b} = \{x \in \mathbb{R}^N : a < |x| < b\}$ with $0 < a < b$ constants in \mathbb{R} , $1 < p \leq N$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $g : [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $g(s) \geq g_0 > 0$.

In order to study the solution of (3.1), one can make a standard change of variables, see Liu and Yang [5]. In the $N \geq p + 1$, if $t = -\frac{A}{r^{(N-p)/(p-1)}} + B$ and $v(t) = u(r)$, where

$$A = \frac{(ab)^{\frac{N-p}{p-1}}}{b^{\frac{N-p}{p-1}} - a^{\frac{N-p}{p-1}}} \quad \text{and} \quad B = \frac{b^{\frac{N-p}{p-1}}}{b^{\frac{N-p}{p-1}} - a^{\frac{N-p}{p-1}}},$$

then the problem (1.1) transforms into the boundary value problem for the nonautonomous ODE

$$(3.2) \quad \begin{cases} (|v'(t)|^{p-2} v'(t))' + h(t)k(t)f(v(t)) = 0 & \text{in } (0, 1), \\ v(0) = v(1) = 0, \end{cases}$$

where

$$h(t) = g\left(\left(\frac{A}{B-t}\right)^{\frac{p-1}{N-p}}\right)$$

and

$$k(t) = \left(\frac{p-1}{N-p}\right)^p \frac{A^{\frac{(p-1)p}{N-p}}}{(B-t)^{\frac{p(N-1)}{N-p}}}.$$

In the case $p = N$, one sets $r = b \left(\frac{a}{b}\right)^t$ and $v(t) = u(r)$, obtaining again the problem (1.2), now with

$$h(t) = g \left(b \left(\frac{a}{b} \right)^t \right) \text{ and } k(t) = \left[b \left(\frac{a}{b} \right)^t \left(\ln \frac{b}{a} \right)^{-1} \right]^{p-1}.$$

Note that in both cases, the function $h(t)$ and $q(t)$ are well defined, continuous and bounded between positive constants in the interval $[0, 1]$, that is, there exists $q_0, q_1, q_2, q_3 > 0$ such that

$$0 < q_0 \leq h(t) \leq q_1 \text{ and } 0 < q_2 \leq k(t) \leq q_3.$$

Then, in (3.1) put $q(t) = h(t)k(t)$ and we can get analogous results as in Theorems 1.1 and 1.3.

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